Acceptance Sampling Procedures for Ballistic Dispersion

W. J. Hurley
Department of Business Administration
Royal Military College of Canada
Kingston, Ontario, Canada. K7K 5L0
613-541-6000 (6468)
613-541-6315
(hurley_w@rmc.ca)

The quality assurance testing of ammunition lots is a critical function of any ammunition procurement program. One important component of these procedures is a test for ballistic dispersion. In the case where dispersion is consistent with a circular normal distribution, this paper argues that the existing tests used by American and Canadian Departments of Defence are not as powerful as they could be. The increased power of the new test is measured using the example of 25mm APFSDS ammunition.

Keywords: Ballistic dispersion, acceptance sampling, 25mm

1. Introduction

An important aspect of ammunition procurement is a series of pre-purchase tests to ensure quality. Substandard ammunition translates to an unacceptable risk for soldiers in the field. The Canadian Department of National Defence (DND) and US Army Ordnance Corp (AOC) do a significant number of tests on all types of ammunition. Among these, there is a test to make sure that the ballistic dispersion of a lot is within an acceptable tolerance.

These tests for dispersion begin by firing a random sample of rounds at a target at a fixed range. The locations of these rounds are recorded and then a mean point of impact is calculated. Given the mean point of impact, the sample standard deviations along each coordinate axis are measured and it is these standard deviations that are used in the test of dispersion. For the lot to pass the test, each sample standard deviation must not exceed some maximum allowable deviation.

While these tests are reasonable, they are not as powerful as they could be. I illustrate this point in the case where the fall of shot follows a circular normal distribution, an assumption widely held for many ballistic rounds. Under this assumption, a uniformly most powerful test of dispersion is characterized by a circular critical region.

To assess the extent of the increase in power the new test provides, the example of 25mm APFSDS (Armour Piercing Fin Stabilized Discarding Sabot) ammunition is considered. The fall of 25mm APFSDS rounds is consistent with a circular normal distribution and so the critical region for a uniformly most powerful test is circular. This circular critical region is shown to give a lower probability of Type II error for values of the underlying dispersion which exceed some predetermined maximum.

Finally I include a short discussion of sample size determination. In tests of ballistic dispersion, there are two important risks. One is that a lot with inferior dispersion is accepted; the other is that a lot with acceptable dispersion is rejected. Clearly the first risk is the more important and the characteristics of the test (critical region and sample size) should take this into account.

The literature on acceptance sampling methods is quite large. Interested readers are referred to [1] for the general theory.

2. Measuring Ballistic Dispersion for 25mm Rounds

The procedures that DND and the AOC use to assess the dispersion of 25mm lots are virtually the same. In the case of the DND test, single rounds are fired from a fixed-mount Mann barrel at a target 300 meters away. The purpose of the Mann barrel is to remove, as much as possible, sources of error other than ballistic error. The locations of these rounds on the target are recorded electronically. Figure 1 shows a sample footprint.

The target is assumed to be infinitely large so that the locations of all rounds fired are known. In some cases, tests of dispersion require that rounds be fired at a finite-sized target. In the case where some rounds miss the target, the analysis would have to be adjusted for censored data. I do not consider this possibility here.

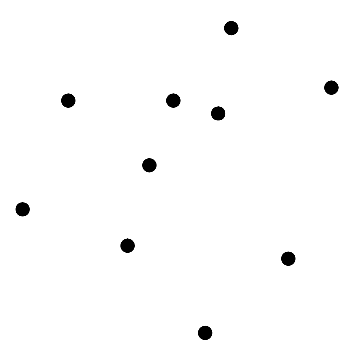

Figure 1. Ballistic Footprint of Ten 25mm Rounds

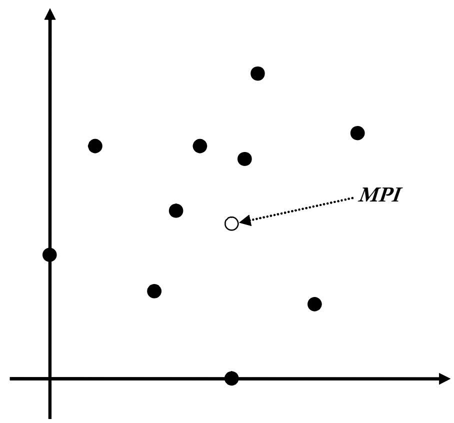

Figure 2. Ballistic Footprint with *X* and *Y* Axes and MPI Shown

To measure dispersion, two axes are drawn by first inserting a y-axis through the point furthest to the left and then an orthogonal x-axis through the point furthest to the bottom. These axes are shown in Figure 2. The locations of the rounds, measured relative to these axes, are presented in Table 1.

Round	*X* (cm)	*Y* (cm)
1	20.5	0.0
2	27.6	7.4
3	13.4	7.8
4	0.0	9.9
5	15.9	19.1
6	6.4	18.7
7	23.0	18.0
8	24.4	22.6
9	24.7	26.5
10	18.0	18.7

Table 1. Locations of the Ten Rounds

More generally, let the location of the points be $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$. The Mean Point of Impact (MPI) is obtained by taking an average of the *x* component of the round locations, and then the *y* components:

$$\bar{x} = \frac{1}{n}\sum_i x_i \text{ and } \bar{y} = \frac{1}{n}\sum_i y_i.$$

The MPI for the dataset in Table 1 is *x* = 17.4 cm and *y* = 14.9 cm.

The measures of dispersion are

$$s_x = \sqrt{\frac{1}{n}\sum_i (x_i - \bar{x})^2} \quad \text{and} \quad s_y = \sqrt{\frac{1}{n}\sum_i (y_i - \bar{y})^2}.$$

These are just the sample standard deviations in each component direction. For the dataset in Table 1, the dispersions are $s_x = 8.3$ cm and $s_y = 7.8$ cm.

These measures have a number of important properties, one of which is that s_x and s_y are independent of the choice of axes. To see this, note that, for the translation $z_i = b + x_i$, we have that

$$\bar{z} = b + \bar{x}$$

and

$$s_z = \sqrt{\frac{1}{n}\sum_i (z_i - \bar{z})^2} = \sqrt{\frac{1}{n}\sum_i (b + x_i - (b + \bar{x}))^2} = s_x.$$

3. Uniformly Most Powerful Tests of Dispersion

Suppose the fall of shot is consistent with a circular normal distribution

$$f_C(x, y) = \frac{1}{2\pi\theta}\exp\left[-\frac{(x-\mu_x)^2 + (y-\mu_y)^2}{2\theta}\right]$$

where the point of aim is (μ_x, μ_y) and the variance is θ. Clearly dispersion is governed by the parameter θ. Suppose that we have picked a random sample of *n* rounds from a lot and these result in the locations

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

that yield the sample standard deviations s_x and s_y. Let us also suppose that we are interested in testing the null hypothesis

$$H_0 : \theta = \theta^*$$

against the alternative

$$H_1 : \theta > \theta^*.$$

We assume that θ^* is the maximum allowable dispersion identified by defence scientists. The existing test of this hypothesis takes the following form:

Step 1: Select a random sample of *n* rounds from the lot, fire them, and observe their resulting locations.

Step 2: Determine whether $\max(s_x, s_y) \le \alpha_0$. If not, the test is failed.

The difficulty is that the non-rejection region does not result in as powerful a test as one based on a uniformly most powerful test. To see this, consider the following argument.

Recalling that the measurement of dispersion is independent of a translation of axes, suppose the coordinate system is chosen with its origin at the MPI. Then the underlying distribution is

$$f_{C_0}(x,y) = \frac{1}{2\pi\theta}\exp\left[-\frac{x^2+y^2}{2\theta}\right].$$

The joint distribution of the sample, then, is

$$J(\theta;(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)) = \left(\frac{1}{2\pi\theta}\right)^n \exp\left[-\frac{1}{2\theta}\left(\sum_i x_i^2 + \sum_i y_i^2\right)\right].$$

Now consider testing

$$H_0 : \theta = \theta^*$$

against the simple alternative

$$H_1 : \theta^{**}$$

where $\theta^{**} > \theta^*$. Then by the Neyman-Pearson lemma, the non-rejection region for a best test is specified by

$$\frac{J(\theta^*;(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n))}{J(\theta^{**};(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n))} \le a$$

where $a > 0$. The ratio on the left-hand side simplifies to

$$\frac{J(\theta^*)}{J(\theta^{**})} = \left(\frac{\theta^{**}}{\theta^*}\right)^n \exp\left[-\frac{1}{2}\left(\frac{\theta^{**}-\theta^*}{\theta^{**}\theta^*}\right)\left(\sum_i x_i^2 + \sum_i y_i^2\right)\right]$$

Taking the ln of both sides of $J(\theta^*)/J(\theta^{**}) \le a,$ gives the critical region

$$\sum_i x_i^2 + \sum_i y_i^2 \geq \left(\frac{2\theta^{**}\theta^{*}}{\theta^{**} - \theta^{*}} \right) \left[n \ln\left(\frac{\theta^{**}}{\theta^{*}} \right) - \ln(a) \right]$$

or

$$\sum_i x_i^2 + \sum_i y_i^2 \geq b$$

where

$$b = \left(\frac{2\theta^{**}\theta^{*}}{\theta^{**} - \theta^{*}} \right) \left[n \ln\left(\frac{\theta^{**}}{\theta^{*}} \right) - \ln(a) \right].$$

But given that the MPI is the origin of the coordinate system, we can rewrite

$$\sum_i x_i^2 + \sum_i y_i^2 \geq b$$

as

$$s_x^2 + s_x^2 \geq b.$$

Under the assumption of the circular normal distribution, the marginal densities in the coordinate directions are normal (see [2, pages 158-159] for a discussion of Hershel's Hypothesis). That is, the density in the x direction is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\theta} \right)$$

and the density in the y direction is

$$f(y) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{y^2}{2\theta} \right).$$

Using this result, we conclude that $\sum_{i=1}^{n} X_i^2 / \theta$ and $\sum_{i=1}^{n} Y_i^2 / \theta$ are both chi-square distributions with n degrees of freedom. Hence we have that

$$\frac{1}{\theta}\left(S_x^2 + S_y^2 \right) = \frac{1}{\theta}\left(\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} Y_i^2 \right)$$

is a chi-square distribution with $n + n = 2n$ degrees of freedom. For a given level of significance, α, we can calculate the value of b / θ using

$$\alpha = \Pr\left(\frac{1}{\theta}\left(\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} Y_i^2\right) \geq \frac{b}{\theta}; H_0\right)$$

and then get b which we re-label b_α. Hence the critical region of size α is

$$s_x^2 + s_x^2 \geq b_\alpha .$$

It is important to point out that the level of significance discussed above is a local measure in the sense that the test for dispersion is not the only test that an ammunition lot would be subjected to. There are a whole series of tests. Consequently the level of significance specified above is not to be confused with the level of significance for the complete battery of tests.

Continuing the argument, there is nothing special about the value θ^{**}. The above argument is true for any value of $\theta^{**} > \theta^{*}$. Consequently, we conclude that $s_x^2 + s_x^2 \geq b_\alpha$ is a uniformly most powerful critical region of size α for testing $H_0 : \theta = \theta^{*}$ against the alternative $H_1 : \theta > \theta^{*}$.

The borders of the critical regions for the existing test and the one developed above are shown in Figure 3. The uniformly most powerful critical region is circular whereas the region for the existing test is rectangular.

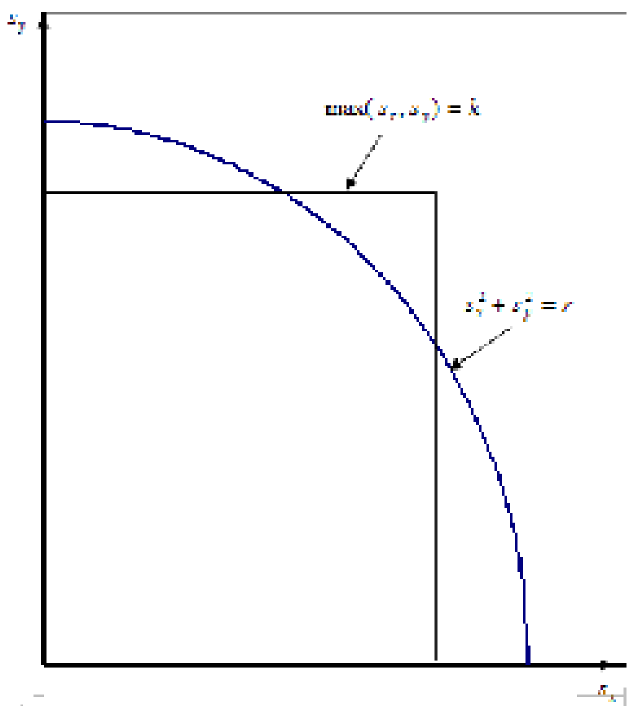

Figure 3. The Comparison of the Critical Regions for the Two Tests

It is worth remarking that this test clarifies the appropriate definition of dispersion. A number of other definitions have been suggested. One of these is a straight average of the Euclidean distances of each sample point to the MPI:

$$D_{xy} = \frac{1}{n}\sum_{i}\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}.$$

In the context of testing ballistic dispersion, the logic above makes it clear that the sample standard deviations in the coordinate directions are the correct way to measure dispersion.

4. An Example

In this section, I assess the increase in power that a uniformly most powerful test will give over the existing test for 25mm APFSDS rounds. For the purposes of distinguishing the two tests, I refer to the existing test as the *Current Test* and the uniformly most powerful test as the *UMP Test*.

To assess the distribution for 25mm APFSDS rounds, I examined the fall of shot for 60 APFSDS rounds. The locations for these rounds are presented in Figure 4. These rounds were fired on three different days, 20 rounds each day. The outside temperature ranged between 5 and 15 degrees Celsius and the wind was negligible, about 2-5 km/hour. I translated each 20-round data set so that its MPI was the origin. The sample standard deviations are $s_x = 8.78$ and $s_y = 8.40$. To test the null that the underlying coordinate dispersions, θ_x and θ_y, are equal, I employed an F-test. The ratio,

$$\frac{s_x^2}{s_y^2} = 1.093,$$

is well within the non-rejection region. Hence the fall of shot for 25mm APFSDS ammunition is consistent with a circular normal distribution.

It is important to clarify that sample size for dispersion testing for 25mm rounds is usually 20 (DND) or 40 (AOC) rounds. The 60 rounds used above was my choice to test whether the fall of 25mm APFSDS rounds is consistent with a circular normal distribution.

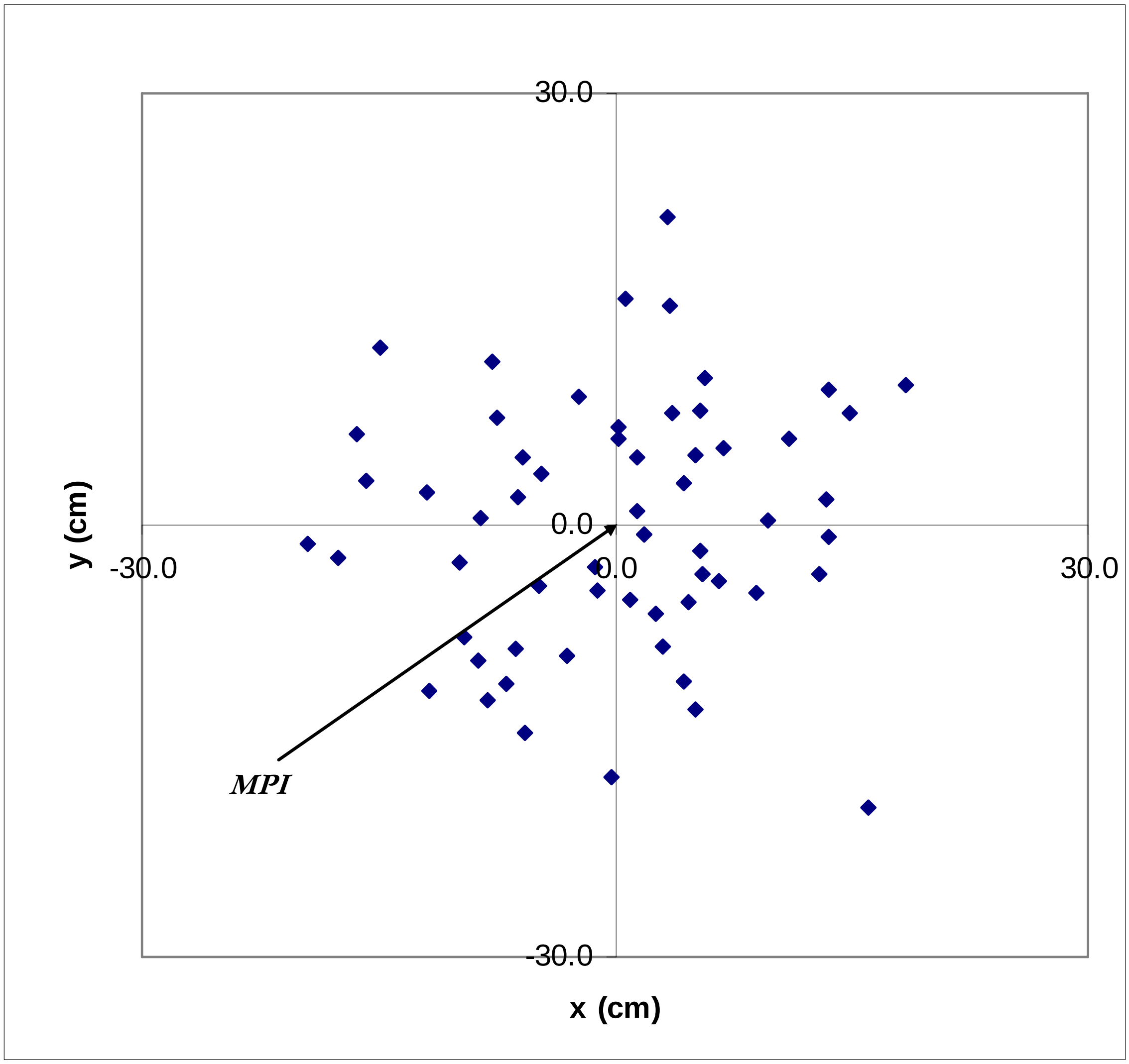

Figure 4. The Fall of Shot for 60 25mm APFSDS Rounds

Suppose that defence scientists have indicated that the dispersion, θ, cannot exceed 10^2, that is

$$\theta \le \theta^* = 10^2.$$

To compare the tests, I fixed the probability of a Type I error (rejecting a lot with underlying dispersion $\theta^* = 10^2$) at 5% for each test and then measured each test's probability of a Type II error (accepting a lot when the dispersion is excessive) for various values of θ which exceed 10^2.

To calculate the critical region for the Current Test for a fixed value of n, I ran 100,000 repetitions of the following experiment:

1. Generate n locations (x_i, y_i) assuming that x_i is normally distributed with mean 0 and standard deviation 10 and y_i is normally distributed with mean 0 and standard deviation 10.
2. Calculate the MPI, $(\bar{x}, \bar{y})$ with $\bar{x} = \sum_i x_i / n$ and $\bar{y} = \sum_i y_i / n.$
3. Calculate the dispersions $s_x = \sqrt{(1/n)\sum_i (x_i - \bar{x})^2}$ and $s_y = \sqrt{(1/n)\sum_i (y_i - \bar{y})^2}$.
4. Calculate $k = \max(s_x, s_y).$

This experiment gives 100,000 observations of $k,$ labeled $k_1, k_2, ..., k_{100,000}.$ Using these, we can then get the value of k^* for which 95% of the observations are less than k^*. Defining the order statistics of $k_1, k_2, ..., k_{100,000}$ to be $k_{(1)}, k_{(2)}, ..., k_{(100,000)},$ we would set $k^* = k_{(95,000)}.$

$k_{(95,000)}$, then, is our estimate of the border of the critical region at a 5% level of significance. To get the critical region for the *UMP Test*, I used tabled values for the chi-square distribution.

Once these critical regions were established, I then executed a similar Monte Carlo experiment to estimate the probabilities of Type II errors. For the *Current Test*, the following steps were executed for specific values of θ which exceeded 10^2:

1. Generate n locations (x_i, y_i) assuming that x_i is normally distributed with mean 0 and variance θ and y_i is normally distributed with mean 0 and variance $\theta.$
2. Calculate the MPI, $(\bar{x}, \bar{y})$ with $\bar{x} = \sum_i x_i / n$ and $\bar{y} = \sum_i y_i / n.$
3. Calculate the dispersions $s_x = \sqrt{(1/n)\sum_i (x_i - \bar{x})^2}$ and $s_y = \sqrt{(1/n)\sum_i (y_i - \bar{y})^2}$.
4. If $\max(s_x, s_y) \le k^*,$ increment a counter by 1, $c(\theta) \leftarrow c(\theta) + 1.$

For a specific value of $\theta,$ this experiment was repeated 100,000 times. The Type II error frequency, then, would be $c(\theta)/100{,}000$ and this is our estimate of the probability of a Type II error for a given $\theta.$

The resulting Type II error frequencies for both tests and $n = 20$ are presented in Table 2. Note that, for each value of $\theta,$ the *UMP Test's* frequency of a Type II error is lower.

For values of θ close to 10^2, these Type II error probabilities are high. One way around this problem is to increase sample size. Type II probability estimates for a sample size $n = 40$ are show in Table 3. Again, the frequencies of Type II errors are lower for the *UMP Test*.

θ	*Current Test*	*UMP Test*
11^2	0.808	0.773
12^2	0.566	0.486
13^2	0.324	0.239
14^2	0.155	0.098
15^2	0.063	0.034

Table 2. Type II Probabilities for a Sample Size of 20

θ	*Current Test*	*UMP Test*
11^2	0.720	0.651
12^2	0.347	0.244
13^2	0.102	0.051
14^2	0.019	0.007
15^2	0.002	0.001

Table 3. Type II Probabilities for a Sample Size of 40

5. Sample Size Considerations

While the main point of this paper is not sample size, it nonetheless bears some discussion. As mentioned in the introduction, there are two important risks in tests of ballistic dispersion. One is that a lot with inferior dispersion is accepted (Soldier Risk); the other is when a lot with acceptable dispersion is rejected (Supplier Risk). Clearly the first risk is the more important one since soldiers in the field could end up with ammunition with inferior ballistic qualities. On the other hand, Supplier Risk is not unimportant. Sending a good quality lot back to the supplier increases the cost of future ammunition since suppliers, quite properly, would include such costs when pricing future lots. Hence the choice of sample size in concert with the critical region should be done in such a way that these two risks are balanced in an appropriate way.

6. Conclusions

In this paper I have argued that the existing tests of ballistic dispersion used by the US and Canadian militaries are not as powerful as they could be. This result is based on the calculation of a uniformly most powerful test when the fall of shot is governed by a circular normal distribution. I then demonstrated the extent of the increase in power that these circular regions provide by considering the case of 25mm APFSDS rounds.

7. References

[1] Stephens, K. S. *The Handbook of Applied Acceptance Sampling: Plans, Procedures and Principles*. ASQ Quality Press, Milwaukee, Wisconsin, 2001.

[2] Rao, C. R. *Linear Statistical Inference and Its Applications*. 2nd edition, John Wiley and Son, New York, 1973.

Author Biography

W. J. Hurley is a Professor of Operations Research at the Royal Military College of Canada. His research interests are in Military Operations Research, Transportation OR, Decision Analysis, Game Theory, and Operations Research in Sports.